

ON THE WELLPOSEDNESS OF THE EXP-RABELO EQUATION

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ABSTRACT. The exp-Rabelo equation describes pseudo-spherical surfaces. It is a nonlinear evolution equation. In this paper the wellposedness of bounded from above solutions for the initial value problem associated to this equation is studied.

1. INTRODUCTION

Bäcklund transformations have been useful in the calculation of soliton solutions of certain nonlinear evolution equations of physical significance [7, 19, 22, 23] restricted to one space variable x and a time coordinate t . The classical treatment of the surface transformations, which provide the origin of Bäcklund theory, was developed in [9]. Bäcklund transformations are local geometric transformations, which construct from a given surface of constant Gaussian curvature -1 a two parameter family of such surfaces. To find such transformations, one needs to solve a system of compatible ordinary differential equations [8].

In [12, 13], the authors used the notion of differential equation for a function $u(t, x)$ that describes a pseudo-spherical surface, and they derived some Bäcklund transformations for nonlinear evolution equations which are the integrability condition $sl(2, R)$ -valued linear problems [11, 10, 15, 16, 23].

In [17], the authors had derived some Bäcklund transformations for nonlinear evolution equations of the AKNS class. These transformations explicitly express the new solutions in terms of the known solutions of the nonlinear evolution equations and corresponding wave functions which are solutions of the associated Ablowitz-Kaup-Newell-Segur (AKNS) system [2, 29].

In [14], the authors used Bäcklund transformations derived in [12, 13] in the construction of exact soliton solutions for some nonlinear evolution equations describing pseudo-spherical surfaces which are beyond the AKNS class. In particular, they analyzed the following equation [3]:

$$(1.1) \quad \partial_x (\partial_t u + \alpha g(u) \partial_x u + \beta \partial_x u) = \gamma g'(u), \quad \alpha, \beta, \gamma \in \mathbb{R},$$

where $g(u)$ is any solution of the linear ordinary differential equation

$$(1.2) \quad g''(u) + \mu g(u) = \theta, \quad \mu, \theta \in \mathbb{R}.$$

(1.1) include the sine-Gordon, sinh-Gordon and Liouville equations, in correspondence of $\alpha = 0$.

In [21], Rabelo proved that the system of the equations (1.1) and (1.2) describes pseudo-spherical surfaces and possesses a zero-curvature representation with a parameter.

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Let us consider (1.1), and assume that $\alpha \neq 0$. In particular, we choose

$$(1.3) \quad \alpha = -1.$$

Taking $\mu = 0$, $\theta = 1$, (1.2) reads

$$(1.4) \quad g''(u) = 1.$$

A solution of (1.4) is

$$(1.5) \quad g(u) = \frac{u^2}{2}.$$

Taking $\beta = 0$, $\gamma > 0$, substituting (1.3), and (1.5) in (1.1), we get

$$(1.6) \quad \partial_x \left(\partial_t u - \frac{1}{6} \partial_x u^3 \right) = \gamma u.$$

(1.6) was also introduced recently by Schäfer and Wayne [26] as a model equation describing the propagation of ultra-short light pulses in silica optical fibers.

Integrating (1.6) in x , we gain the integro-differential formulation of (1.6) (see [24])

$$(1.7) \quad \partial_t u - \frac{1}{6} \partial_x u^3 = \gamma \int^x u(t, y) dy,$$

that is equivalent to

$$(1.8) \quad \partial_t u - \frac{1}{6} \partial_x u^3 = \gamma P, \quad \partial_x P = u.$$

In [4, 5, 6], the authors investigated the well-posedness in classes of discontinuous functions for (1.7), or (1.8). In particular, they proved that (1.7), or (1.8) admits an unique entropy solution in the sense of the following definition:

Definition 1.1. *We say that $u \in L^\infty((0, T) \times \mathbb{R})$, $T > 0$, is an entropy solution of (1.7), or (1.8) if*

- i) u is a distributional solution of (1.7) or equivalently of (1.8);*
- ii) for every convex function $\eta \in C^2(\mathbb{R})$ the entropy inequality*

$$(1.9) \quad \partial_t \eta(u) + \partial_x q(u) - \gamma \eta'(u) P \leq 0, \quad q(u) = - \int^u \frac{\xi^2}{2} \eta'(\xi) d\xi,$$

holds in the sense of distributions in $(0, \infty) \times \mathbb{R}$.

Here, we consider the case

$$(1.10) \quad \alpha = 1.$$

Taking $\mu = -1$, $\theta = 0$, (1.2) reads

$$(1.11) \quad g''(u) - g(u) = 0.$$

A solution of (1.11) is

$$(1.12) \quad g(u) = e^u.$$

Taking $\beta = 0$, $\gamma = -1$, and substituting (1.10), and (1.12) in (1.1), we get

$$(1.13) \quad \partial_x (\partial_t u + \partial_x e^u) = -e^u,$$

which is known as the exp-Rabelo equation (see [25]).

Our aim is to investigate the well-posedness for the initial value problem in classes of discontinuous functions for (1.13). Therefore, we augment (1.13) with the initial datum

$$(1.14) \quad u(0, x) = u_0(x),$$

on which we assume that

$$(1.15) \quad \sup u_0 < \infty, \quad \int_{\mathbb{R}} e^{u_0(x)} dx < \infty.$$

Integrating (1.13) in $(0, x)$ we gain the integro-differential formulation of (1.13) (see [1, 25, 28])

$$(1.16) \quad \begin{cases} \partial_t u + \partial_x e^u = - \int_0^x e^{u(t,y)} dy, & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

that is equivalent to

$$(1.17) \quad \begin{cases} \partial_t u + \partial_x e^u = -P, & t > 0, x \in \mathbb{R}, \\ \partial_x P = e^u, & t > 0, x > 0, \\ P(t, 0) = 0, & t > 0, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

We give the following definition of solution:

Definition 1.2. We say that u , such that

$$(1.18) \quad \sup u(t, \cdot) < \infty, \quad \int_{\mathbb{R}} e^{u(t,x)} dx < \infty, \quad t > 0,$$

is an entropy solution of the initial value problem (1.13) and (1.14) if

- i) u is a distributional solution of (1.16) or equivalently of (1.17);
- ii) for every convex function $\eta \in C^2(\mathbb{R})$ the entropy inequality

$$(1.19) \quad \partial_t \eta(u) + \partial_x q(u) + \eta'(u) \int_0^x e^u dy \leq 0, \quad q(u) = \int^u e^\xi \eta'(\xi) d\xi,$$

holds in the sense of distributions in $(0, \infty) \times \mathbb{R}$.

The main result of this paper is the following theorem.

Theorem 1.1. Let $T > 0$. Assume (1.15). The initial value problem (1.13) and (1.14) possesses an unique entropy solution u in the sense of Definition 1.2. Moreover, if u and w are two entropy solutions of (1.13) and (1.14) in the sense of Definition 1.2 the following inequality holds

$$(1.20) \quad \|u(t, \cdot) - w(t, \cdot)\|_{L^1(-R, R)} \leq e^{C(T)t} \|u(0, \cdot) - w(0, \cdot)\|_{L^1(-R-C(T)t, R+C(T)t)},$$

for almost every $0 < t < T$, $R > 0$, and some suitable constant $C(T) > 0$.

The paper is organized as follows. In Section 2 we prove several a priori estimates on a vanishing viscosity approximation of (1.17). Those play a key role in the proof of our main result, that is given in Section 3.

2. VANISHING VISCOSITY APPROXIMATION

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (1.17).

Fix a small number $\varepsilon > 0$, and let $u_\varepsilon = u_\varepsilon(t, x)$ be the unique classical solution of the following mixed problem

$$(2.1) \quad \begin{cases} \partial_t u_\varepsilon + \partial_x e^{u_\varepsilon} = -P_\varepsilon + \varepsilon \partial_{xx}^2 (e^{u_\varepsilon}), & t > 0, x \in \mathbb{R}, \\ \partial_x P_\varepsilon = e^{u_\varepsilon}, & t > 0, x \in \mathbb{R}, \\ P_\varepsilon(t, 0) = 0, & t > 0, \\ u_\varepsilon(0, x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases}$$

where $u_{\varepsilon,0}$ is $C^\infty(0, \infty)$ approximations of u_0 such that

$$(2.2) \quad \begin{aligned} u_{0,\varepsilon} &\rightarrow u_0, \quad \text{a.e. and in } L_{loc}^p(\mathbb{R}), \quad 1 \leq p < \infty, \\ \sup u_{\varepsilon,0} &\leq \sup u_0, \quad \varepsilon > 0, \\ \int_{\mathbb{R}} e^{u_{\varepsilon,0}(x)} dx &\leq \int_{\mathbb{R}} e^{u_0(x)} dx, \quad \varepsilon > 0. \end{aligned}$$

Clearly, (2.1) is equivalent to the integro-differential problem

$$(2.3) \quad \begin{cases} \partial_t u_\varepsilon + \partial_x e^{u_\varepsilon} = - \int_0^x e^{u_\varepsilon(t,y)} dy + \varepsilon \partial_{xx}^2 (e^{u_\varepsilon}), & t > 0, \quad x \in \mathbb{R}, \\ u_\varepsilon(0, x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}. \end{cases}$$

Observe that, multiplying (2.3) by $e^{u_\varepsilon(t,x)}$, we have

$$(2.4) \quad \partial_t (e^{u_\varepsilon}) + \frac{1}{2} \partial_x (e^{2u_\varepsilon}) = -e^{u_\varepsilon} \int_0^x e^{u_\varepsilon(t,y)} dy + \varepsilon e^{u_\varepsilon} \partial_{xx}^2 (e^{u_\varepsilon}).$$

Introducing the notation

$$(2.5) \quad v_\varepsilon(t, x) = e^{u_\varepsilon(t,x)} > 0,$$

(2.4) reads

$$(2.6) \quad \partial_t v_\varepsilon + \frac{1}{2} \partial_x v_\varepsilon^2 = -v_\varepsilon \int_0^x v_\varepsilon(t, y) dy + \varepsilon v_\varepsilon \partial_{xx}^2 v_\varepsilon.$$

It follows from (2.5) and $u_\varepsilon(t, \pm\infty) = -\infty$ that

$$(2.7) \quad v_\varepsilon(t, \pm\infty) = 0.$$

Moreover, from (2.2) and (2.5), we get

$$(2.8) \quad \|v_{\varepsilon,0}\|_{L^\infty(\mathbb{R})} \leq e^{\sup u_0}, \quad \int_{\mathbb{R}} v_{\varepsilon,0}(x) dx \leq \int_{\mathbb{R}} e^{u_0(x)} dx, \quad \varepsilon > 0.$$

Let us prove some a priori estimates on v_ε , and, hence on u_ε .

Lemma 2.1. *We have that*

$$(2.9) \quad \|v_\varepsilon\|_{L^\infty((0,\infty) \times \mathbb{R})} \leq e^{\sup u_0}, \quad \varepsilon > 0.$$

In particular, we get

$$(2.10) \quad \sup u_\varepsilon(t, \cdot) \leq \sup u_0, \quad t > 0.$$

Proof. We begin by observing that, from (2.5) and (2.6), we have

$$(2.11) \quad \partial_t v_\varepsilon + \partial_x \left(\frac{v_\varepsilon^2}{2} \right) - \varepsilon \partial_{xx}^2 \left(\frac{v_\varepsilon^2}{2} \right) \leq 0.$$

Therefore, a supersolution of (2.6) satisfies the following ordinary differential equation

$$\frac{dz}{dt} = 0, \quad z(0) = e^{\sup u_0},$$

that is

$$(2.12) \quad z(t) = e^{\sup u_0}.$$

It follows from the comparison principle for parabolic equation and (2.5) that

$$(2.13) \quad 0 < v_\varepsilon(t, x) \leq e^{\sup u_0},$$

which gives (2.9).

Finally, (2.10) follows from (2.5) and (2.13). □

Lemma 2.2. *Let $\alpha \geq 0$. For each $t > 0$, we have*

$$(2.14) \quad \begin{aligned} & \int_{\mathbb{R}} v_{\varepsilon}^{\alpha+1} + \varepsilon(\alpha+1)^2 \int_0^t \int_{\mathbb{R}} v_{\varepsilon}^{\alpha} (\partial_x v_{\varepsilon})^2 ds dx \\ & + (\alpha+1) \int_0^t \int_{\mathbb{R}} v_{\varepsilon}^{\alpha+1} \left(\int_0^x v_{\varepsilon} dy \right) ds dx \leq (e^{\sup u_0})^{\alpha} \int_{\mathbb{R}} e^{u_0(x)} dx. \end{aligned}$$

In particular, we get

$$(2.15) \quad \begin{aligned} & \int_{\mathbb{R}} e^{(\alpha+1)u_{\varepsilon}(t,x)} dx \leq (e^{\sup u_0})^{\alpha} \int_{\mathbb{R}} e^{u_0(x)} dx, \\ & \varepsilon(\alpha+1)^2 \int_0^t \int_{\mathbb{R}} e^{\alpha u_{\varepsilon}(t,x)} \left(\partial_x e^{u_{\varepsilon}(t,x)} \right)^2 ds dx \leq (e^{\sup u_0})^{\alpha} \int_{\mathbb{R}} e^{u_0(x)} dx, \\ & (\alpha+1) \int_0^t \int_{\mathbb{R}} e^{(\alpha+1)u_{\varepsilon}(t,x)} \left(\int_0^x e^{u_{\varepsilon}(t,x)} dy \right) ds dx \leq (e^{\sup u_0})^{\alpha} \int_{\mathbb{R}} e^{u_0(x)} dx. \end{aligned}$$

Proof. Multiplying (2.6) by v_{ε}^{α} , we have

$$v_{\varepsilon}^{\alpha} \partial_t v_{\varepsilon} + v_{\varepsilon}^{\alpha+1} \partial_x v_{\varepsilon} = -v_{\varepsilon}^{\alpha+1} \int_0^x v_{\varepsilon} dy + \varepsilon v_{\varepsilon}^{\alpha+1} \partial_{xx}^2 v_{\varepsilon}.$$

It follows from (2.5), (2.6) and an integration on \mathbb{R} that

$$\begin{aligned} \frac{1}{\alpha+1} \frac{d}{dt} \int_{\mathbb{R}} v_{\varepsilon}^{\alpha+1} &= \int_{\mathbb{R}} v_{\varepsilon}^{\alpha} \partial_t v_{\varepsilon} \\ &= \varepsilon \int_{\mathbb{R}} v_{\varepsilon}^{\alpha+1} \partial_{xx}^2 v_{\varepsilon} dx - \int_{\mathbb{R}} v_{\varepsilon}^{\alpha+1} \partial_x v_{\varepsilon} - \int_{\mathbb{R}} v_{\varepsilon}^{\alpha+1} \left(\int_0^x v_{\varepsilon} dy \right) dx \\ &= -\varepsilon(\alpha+1) \int_{\mathbb{R}} v_{\varepsilon}^{\alpha} (\partial_x v_{\varepsilon})^2 dx - \int_{\mathbb{R}} v_{\varepsilon}^{\alpha+1} \left(\int_0^x v_{\varepsilon} dy \right) dx, \end{aligned}$$

that is,

$$(2.16) \quad \frac{d}{dt} \int_{\mathbb{R}} v_{\varepsilon}^{\alpha+1} + \varepsilon(\alpha+1)^2 \int_{\mathbb{R}} v_{\varepsilon}^{\alpha} (\partial_x v_{\varepsilon})^2 dx + (\alpha+1) \int_{\mathbb{R}} v_{\varepsilon}^{\alpha+1} \left(\int_0^x v_{\varepsilon} dy \right) dx = 0.$$

An integration on $(0, t)$ gives

$$(2.17) \quad \begin{aligned} & \int_{\mathbb{R}} v_{\varepsilon}^{\alpha+1} dx + \varepsilon(\alpha+1)^2 \int_0^t \int_{\mathbb{R}} v_{\varepsilon}^{\alpha} (\partial_x v_{\varepsilon})^2 ds dx \\ & + (\alpha+1) \int_0^t \int_{\mathbb{R}} v_{\varepsilon}^{\alpha+1} \left(\int_0^x v_{\varepsilon} dy \right) dx = \int_{\mathbb{R}} v_{\varepsilon,0}^{\alpha+1} dx. \end{aligned}$$

From (2.5) and (2.8),

$$(2.18) \quad \int_{\mathbb{R}} v_{\varepsilon,0}^{\alpha+1} dx \leq \|v_{\varepsilon,0}\|_{L^{\infty}((0,\infty) \times \mathbb{R})}^{\alpha} \int_{\mathbb{R}} v_{\varepsilon,0} dx \leq (e^{\sup u_0})^{\alpha} \int_{\mathbb{R}} e^{u_0(x)} dx.$$

Therefore, (2.17) and (2.18) give (2.14).

Finally, (2.15) follows from (2.6) and (2.14) □

3. PROOF OF THEOREM 1.1

This section is devoted to the proof of Theorem 1.1. We begin with the following result

Lemma 3.1. *Let $T > 0$. There exists a subsequence $\{v_{\varepsilon_k}\}_{k \in \mathbb{N}}$ of $\{v_{\varepsilon}\}_{\varepsilon > 0}$ and a limit function $v \in L^{\infty}((0, \infty) \times \mathbb{R})$ such that*

$$(3.1) \quad v_{\varepsilon_k} \rightarrow v \text{ a.e. and in } L_{loc}^p((0, \infty) \times \mathbb{R}), \quad 1 \leq p < \infty.$$

In particular, we have

$$(3.2) \quad u_{\varepsilon_k} \rightarrow \log v = u \text{ a.e. and in } L_{loc}^p((0, \infty) \times \mathbb{R}), \quad 1 \leq p < \infty$$

Proof. Let $\eta : \mathbb{R} \rightarrow \mathbb{R}$ be any convex C^2 entropy function, and $q : \mathbb{R} \rightarrow \mathbb{R}$ be the corresponding entropy flux defined by $q'(v) = v\eta'(v)$. By multiplying (2.6) with $\eta'(v_\varepsilon)$ and using the chain rule, we get

$$\begin{aligned} \partial_t \eta(v_\varepsilon) + \partial_x q(v_\varepsilon) &= \underbrace{\varepsilon \partial_x (\eta'(v_\varepsilon) v_\varepsilon \partial_x v_\varepsilon)}_{=: \mathcal{L}_{1,\varepsilon}} - \underbrace{\varepsilon \eta''(v_\varepsilon) v_\varepsilon (\partial_x u_\varepsilon)^2}_{=: \mathcal{L}_{2,\varepsilon}} \\ &\quad - \underbrace{\varepsilon \eta'(v_\varepsilon) (\partial_x v_\varepsilon)^2}_{=: \mathcal{L}_{3,\varepsilon}} - \underbrace{\eta'(v_\varepsilon) v_\varepsilon \int_0^x v_\varepsilon dy}_{=: \mathcal{L}_{4,\varepsilon}}, \end{aligned}$$

where $\mathcal{L}_{1,\varepsilon}$, $\mathcal{L}_{2,\varepsilon}$, $\mathcal{L}_{3,\varepsilon}$, $\mathcal{L}_{4,\varepsilon}$ are distributions. Let us show that

$$\mathcal{L}_{1,\varepsilon} \rightarrow 0 \text{ in } H^{-1}((0, T) \times \mathbb{R}), \quad T > 0.$$

By Lemmas 2.1 and 2.2 in correspondence of $\alpha = 0$,

$$\begin{aligned} \|\varepsilon \eta'(v_\varepsilon) v_\varepsilon \partial_x v_\varepsilon\|_{L^2((0,T) \times \mathbb{R})}^2 &\leq \varepsilon^2 \|\eta'\|_{L^\infty(I)}^2 \|v_\varepsilon\|_{L^\infty((0,\infty) \times \mathbb{R})} \int_0^T \|\partial_x v_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ &\leq \varepsilon \|\eta'\|_{L^\infty(T)}^2 e^{\sup u_0} \int_{\mathbb{R}} e^{u_0(x)} dx \rightarrow 0, \end{aligned}$$

where

$$I = (0, e^{\sup u_0}).$$

We claim that

$$\{\mathcal{L}_{2,\varepsilon}\}_{\varepsilon>0} \text{ is uniformly bounded in } L^1((0, T) \times \mathbb{R}), \quad T > 0.$$

Again by Lemmas 2.1 and 2.2 in correspondence of $\alpha = 0$,

$$\begin{aligned} \|\varepsilon \eta''(v_\varepsilon) v_\varepsilon (\partial_x v_\varepsilon)^2\|_{L^1((0,T) \times \mathbb{R})} &\leq \|\eta''\|_{L^\infty(I)} \|v_\varepsilon\|_{L^\infty((0,\infty) \times \mathbb{R})} \varepsilon \int_0^T \|\partial_x v_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ &\leq \|\eta''\|_{L^\infty(I)} e^{\sup u_0} \int_{\mathbb{R}} e^{u_0(x)} dx. \end{aligned}$$

We have that

$$\{\mathcal{L}_{3,\varepsilon}\}_{\varepsilon>0} \text{ is uniformly bounded in } L^1((0, T) \times \mathbb{R}), \quad T > 0.$$

Again by Lemmas 2.1 and 2.2 in correspondence of $\alpha = 0$,

$$\begin{aligned} \|\varepsilon \eta'(v_\varepsilon) (\partial_x v_\varepsilon)^2\|_{L^1((0,T) \times \mathbb{R})} &\leq \|\eta'\|_{L^\infty(I)} \varepsilon \int_0^T \|\partial_x v_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ &\leq \|\eta'\|_{L^\infty(I)} \int_{\mathbb{R}} e^{u_0(x)} dx. \end{aligned}$$

We claim that

$$\{\mathcal{L}_{4,\varepsilon}\}_{\varepsilon>0} \text{ is uniformly bounded in } L^1((0, T) \times \mathbb{R}), \quad T > 0.$$

Again by Lemmas 2.1 and 2.2 in correspondence of $\alpha = 0$,

$$\begin{aligned} \left\| \eta'(v_\varepsilon) v_\varepsilon \int_0^x v_\varepsilon dy \right\|_{L^1((0,T) \times \mathbb{R})} &\leq \|\eta'\|_{L^\infty(I)} \int_0^T \int_{\mathbb{R}} v_\varepsilon \left(\int_0^x v_\varepsilon dy \right) ds dx \\ &\leq \|\eta'\|_{L^\infty(I)} \int_{\mathbb{R}} e^{u_0(x)} dx. \end{aligned}$$

Therefore, Murat's lemma [20] implies that

$$(3.3) \quad \{\partial_t \eta(v_\varepsilon) + \partial_x q(v_\varepsilon)\}_{\varepsilon > 0} \text{ lies in a compact subset of } H_{\text{loc}}^{-1}((0, T) \times \mathbb{R}).$$

The L^∞ bound stated in Lemma 2.1, (3.3), and the Tartar's compensated compactness method [27] give the existence of a subsequence $\{v_{\varepsilon_k}\}_{k \in \mathbb{N}}$ and a limit function $v \in L^\infty((0, \infty) \times \mathbb{R})$, such that (3.1) holds.

(3.2) follows from (2.5) and (3.1). \square

To prove Theorem 1.1, we consider the following definition.

Definition 3.1. A pair of functions (η, q) is called an entropy-entropy flux pair if $\eta : \mathbb{R} \rightarrow \mathbb{R}$ is a C^2 function and $q : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$q(u) = \int_0^u \eta'(\xi) f'(\xi) d\xi.$$

An entropy-entropy flux pair (η, q) is called convex/compactly supported if, in addition, η is convex/compactly supported.

Proof of Theorem 1.1. We begin by proving that u , defined in (3.2), is an entropy solution of (1.16), or (1.17) in the sense of Definition 1.2. Let $\phi \in C^\infty(\mathbb{R}^2)$ be a positive test function with a support, and let us consider a compactly supported entropy-entropy flux pair (η, q) . We have to prove

$$(3.4) \quad \int_0^\infty \int_{\mathbb{R}} (\eta(u) \partial_t \phi + q(u) \partial_x \phi) dt dx - \int_0^\infty \int_{\mathbb{R}} \eta'(u) \left(\int_0^x e^u dy \right) dt dx + \int_{\mathbb{R}} \eta(u_0(x)) \phi(0, x) dx \geq 0.$$

Multiplying (2.1) by $\eta'(u_\varepsilon)$, we have

$$\partial_t \eta(u_{\varepsilon_k}) + \partial_x q(u_{\varepsilon_k}) + \eta'(u_{\varepsilon_k}) \int_0^x e^{u_{\varepsilon_k}} dy = \varepsilon_k \eta'(u_{\varepsilon_k}) \partial_{xx}^2 (e^{u_{\varepsilon_k}}).$$

Since

$$\begin{aligned} \varepsilon_k \eta'(u_{\varepsilon_k}) \partial_{xx}^2 (e^{u_{\varepsilon_k}}) &= \partial_x (\varepsilon_k \eta'(u_{\varepsilon_k}) \partial_x (e^{u_{\varepsilon_k}})) - \varepsilon_k \eta''(u_{\varepsilon_k}) \partial_x (e^{u_{\varepsilon_k}}) \partial_x u_{\varepsilon_k} \\ &= \partial_x (\varepsilon_k \eta'(u_{\varepsilon_k}) \partial_x (e^{u_{\varepsilon_k}})) - \varepsilon_k \eta''(u_{\varepsilon_k}) e^{u_{\varepsilon_k}} (\partial_x u_{\varepsilon_k})^2, \end{aligned}$$

we have

$$(3.5) \quad \begin{aligned} &\partial_t \eta(u_{\varepsilon_k}) + \partial_x q(u_{\varepsilon_k}) + \eta'(u_{\varepsilon_k}) \int_0^x e^{u_{\varepsilon_k}} dy \\ &= \partial_x (\varepsilon_k \eta'(u_{\varepsilon_k}) \partial_x (e^{u_{\varepsilon_k}})) - \varepsilon_k \eta''(u_{\varepsilon_k}) e^{u_{\varepsilon_k}} (\partial_x u_{\varepsilon_k})^2 \\ &\leq \partial_x (\varepsilon_k \eta'(u_{\varepsilon_k}) \partial_x (e^{u_{\varepsilon_k}})). \end{aligned}$$

Multiplying (3.5) by ϕ , an integration on $(0, \infty) \times \mathbb{R}$ gives

$$(3.6) \quad \begin{aligned} &\int_0^\infty \int_{\mathbb{R}} (\eta(u_{\varepsilon_k}) \partial_t \phi + q(u_{\varepsilon_k}) \partial_x \phi) dt dx - \int_0^\infty \int_{\mathbb{R}} \eta'(u_{\varepsilon_k}) \left(\int_0^x e^{u_{\varepsilon_k}} dy \right) dt dx \\ &+ \int_{\mathbb{R}} \eta(u_{\varepsilon_k, 0}(x)) \phi(0, x) dx \\ &+ \varepsilon_k \int_0^\infty \int_{\mathbb{R}} \eta'(u_{\varepsilon_k}) \partial_x (e^{u_{\varepsilon_k}}) \partial_x \phi dt dx \geq 0. \end{aligned}$$

Let us show that

$$(3.7) \quad \varepsilon_k \int_0^\infty \int_{\mathbb{R}} \eta'(u_{\varepsilon_k}) \partial_x (e^{u_{\varepsilon_k}}) \partial_x \phi dt dx \rightarrow 0.$$

Fix $T > 0$. From (2.14) in correspondence of $\alpha = 0$, and the Hölder inequality,

$$\begin{aligned}
& \varepsilon_k \int_0^\infty \int_{\mathbb{R}} \eta'(u_{\varepsilon_k}) \partial_x (e^{u_{\varepsilon_k}}) \partial_x \phi dt dx \\
& \leq \varepsilon_k \int_0^\infty \int_{\mathbb{R}} |\eta'(u_{\varepsilon_k})| |\partial_x (e^{u_{\varepsilon_k}})| |\partial_x \phi| dt dx \\
& \leq \varepsilon_k \|\eta'\|_{L^\infty((0,\infty) \times \mathbb{R})} \|\partial_x (e^{u_{\varepsilon_k}})\|_{L^2(\text{supp}(\partial_x \phi))} \|\partial_x \phi\|_{L^2(\text{supp}(\partial_x \phi))} \\
& \leq \varepsilon_k \|\eta'\|_{L^\infty((0,\infty) \times \mathbb{R})} \|\partial_x (e^{u_{\varepsilon_k}})\|_{L^2((0,T) \times \mathbb{R})} \|\partial_x \phi\|_{L^2((0,T) \times \mathbb{R})} \\
& \leq \sqrt{\varepsilon_k} \|\eta'\|_{L^\infty((0,\infty) \times \mathbb{R})} \|\partial_x \phi\|_{L^2((0,T) \times \mathbb{R})} \left(\int_{\mathbb{R}} e^{u_0(x)} dx \right)^{\frac{1}{2}} \rightarrow 0,
\end{aligned}$$

that is (3.7).

Therefore, (3.6) follows from (2.2), (3.2), (3.6) and (3.7).

Let us prove that $u(t, x)$ is unique and (1.20) holds. Assume that $u(t, x)$, $w(t, x)$ satisfy

$$\begin{aligned}
& \sup u(t, \cdot) \leq \sup u_0, \quad \sup w(t, \cdot) \leq \sup w_0, \quad t > 0, \\
(3.8) \quad & \int_{\mathbb{R}} e^{u_0(x)} dx < \infty, \quad \int_{\mathbb{R}} e^{w_0(x)} dx < \infty,
\end{aligned}$$

two entropy solutions of (1.16), or (1.17). Due to (3.8), we have

$$(3.9) \quad |e^u - e^w| \leq C_0 |u - w|,$$

where

$$C_0 = \sup_{\mathbb{R}} \{e^{\sup u_0} + e^{\sup w_0}\}.$$

Arguing as in [5, 6, 18], we can prove that

$$\partial_t(|u - w|) + \partial_x((e^u - e^w) \text{sign}(u - w)) + \text{sign}(u - w) \int_0^x (e^u - e^w) dy \leq 0$$

holds in sense of distributions in $(0, \infty) \times \mathbb{R}$, and

$$\begin{aligned}
(3.10) \quad & \|u(t, \cdot) - w(t, \cdot)\|_{I(t)} \\
& \leq \|u_0 - w_0\|_{I(0)} - \int_0^t \int_{I(s)} \text{sign}(u - w) \left(\int_0^x (e^u - e^w) dy \right) ds dx, \quad 0 < t < T,
\end{aligned}$$

where

$$I(s) = [-R - C_0(t - s), R + C_0(t - s)].$$

Due to (3.9),

$$\begin{aligned}
(3.11) \quad & - \int_0^t \int_{I(s)} \text{sign}(u - v) \left(\int_0^x (e^u - e^w) dy \right) ds dx \\
& \leq \int_0^t \int_{I(s)} \left| \left(\int_0^x |e^u - e^w| dy \right) \right| ds dx \\
& \leq C_0 \int_0^t \int_{I(s)} \left(\left| \int_{I(s)} |u - v| dy \right| \right) ds dx \\
& = C_0 \int_0^t |I(s)| \|u(s, \cdot) - v(s, \cdot)\|_{L^1(I(s))} ds.
\end{aligned}$$

Moreover,

$$(3.12) \quad |I(s)| = 2R + 2C_0(t - s) \leq 2R + 2C_0t \leq 2R + 2C_0T.$$

We consider the following continuous function:

$$(3.13) \quad G(t) = \|u(t, \cdot) - v(t, \cdot)\|_{L^1(I(t))}, \quad t \geq 0.$$

It follows from (3.10), (3.11), (3.12) and (3.13) that

$$G(t) \leq G(0) + C(T) \int_0^t G(s) ds,$$

where $C(T) = 2R + 2C_0T$. The Gronwall inequality and (3.13) give

$$\|u(t, \cdot) - v(t, \cdot)\|_{L^1(-R, R)} \leq e^{C(T)t} \|u_0 - v_0\|_{L^1(-R-C_0t, R+C_0t)},$$

that is (1.20). \square

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